

Entanglement in Coupled Harmonic Oscillators Studied Using a Unitary Transformation

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Abstract

We develop an approach to study the entanglement in two coupled harmonic oscillators. We start by introducing an unitary transformation to end up with the solutions of the energy spectrum. These are used to construct the corresponding coherent states through the standard way. To evaluate the degree of the entanglement between the obtained states, we calculate the purity function in terms of the coherent and number states, separately. The result is yielded two parameters dependance of the purity, which can be controlled easily. Interesting results are derived by fixing the mixing angle of such transformation as $\frac{\pi}{2}$. We compare our results with already published work and point out the relevance of these findings to a systematic formulation of the entanglement effect in two coupled harmonic oscillators.

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1 Introduction

Entanglement is one of the most remarkable features of quantum mechanics that does not have any classical counterpart. It is a notion which has been initially introduced and coined by Schrödinger [1] when quantum mechanics was still in its early stage of development. Its status has evolved throughout the decades and has been subjected to significant changes. Traditionally, entanglement has been related to the most quantum mechanical exotic concepts such as Schrödinger cat [1], Einstein-Podolsky-Rosen paradox [2] and violation of Bell's inequalities [3]. Despite its conventional significance, entanglement has gained, in the last decades, a renewed interest mainly because of the development of the quantum information science [4]. It has been revealed that it lies at the heart of various communication and computational tasks that cannot be implemented classically. It is believed that entanglement is the main ingredient of the quantum speed-up in quantum computation [4]. Moreover, several quantum protocols such as teleportation, quantum dense coding, and so on [5, 6, 7, 8, 9, 10, 11] are exclusively realized with the help of entangled states. With this respect, many interesting works appeared dealing with the development of a quantitative theory of entanglement and the definition of its basic measure. These concern the concurrence, entanglement of formation and linear entropy [12, 13, 14, 15].

Entangled quantum systems can exhibit correlations that cannot be explained on the basis of classical laws and the entanglement in a collection of states is clearly a signature of non-classicality [16]. Furthermore, in the last few years it has become evident that quantum information may lead to further insights into other areas of physics [17]. This has led to a cross-fertilizing between different areas of physics. It is worthy of note that the nonlinear Kerr effect [18] has been considered as the most famous source of physical realization of photon pairs of entangled polarization states. However, it raises a number of difficulties to the control of photons that are traveling at the speed of light. This is why so much attention has been paid recently to the entangled states of massive particles as they are viewed to be much more easy to control [17, 19].

On the other hand, the harmonic oscillator machinery plays a crucial role in many areas of physics. These are the Lee model in quantum field theory [20], the Bogoliubov transformation in superconductivity [21], two-mode squeezed states of light [22, 23, 24], the covariant harmonic oscillator model for the parton picture [25], and models in molecular physics [26]. There are also models in which one of the variables is not observed, including thermo-field dynamics [27], two-mode squeezed states [28, 29], the hadronic temperature [30], and the Barnett-Phoenix version of information theory [31]. These physical models are the examples of Feynman's rest of the universe. In the case of two coupled harmonic oscillator, the first one is the universe and the second one is the rest of universe. For sake of the mathematical simplicity, the mixing angle (rotation of the coordinate system), in the above mentioned references, is taken to be equal $\frac{\pi}{2}$. This means that the system consists of two identical oscillator coupled together by a potential term.

In the context of the entangled massive particles, we cite the recently achieved investigation of a specific realization of two coupled harmonic oscillator model by the authors of reference [19]. In fact, they calculated the interatomic entanglement for Gaussian and non-Gaussian pure states by using the purity function of the reduced density matrix. This allowed them to treat the cases of free and trapped molecules and hetero- and homonuclear molecules. Finally, they concluded that when the trap frequency and the molecular frequency are very different, and when the atomic masses are equal,

the atoms are highly-entangled for molecular coherent states and number states. Surprisingly, while the interatomic entanglement can be quite large even for molecular coherent states, the covariance of atomic position and momentum observables can be entirely explained by a classical model with appropriately chosen statistical uncertainty.

Motivated by the mentioned references above and in particular [19], we undertake to develop a new approach to study the entanglement in two coupled harmonic oscillators. It is based on a suitable transformation having the merit of reducing the relevant physical parameters into two: the coupling parameter η and mixing angle θ . It turns out that we can easily derive the solutions corresponding to the energy spectrum. Then, the obtained solutions are used to construct the coherent states through the standard method. In order to characterize the degree of entanglement, we calculate, within the framework of the coherent states, the purity function. Then the final form of the purity is cast in terms of η and θ . Our finding shows two interesting results: the first one tells us that the present system is not entangled at $\eta = 0$, as expected, and highly entangled at large η (Figure 1). The second one is when we fix $\theta = \frac{\pi}{2}$, the purity behaves like the inverse of $\cosh \eta$ and the corresponding plot (Figure 2) shows that the purity is ranging between 0 and 1. It is worthy of notice that, in this case, the purity becomes one parameter dependent, which means that it is easy to control.

Subsequently, we evaluate the purity in terms of the number states. In doing so, we use the well-known relation to express the number states $|n_1, n_2\rangle$ as function of the corresponding coherent states $|\alpha, \beta\rangle$. Then after a lengthy but a straightforward algebra, we end up with the final form of purity. To be much more concrete, we restrict ourselves to some interesting cases that are $(n_1 = 1, n_2 = 0)$ and $(n_1 = 1, n_2 = 1)$. For the first configuration, the obtained purity is simply a ratio of hyperbolic and sinusoidal functions, which tells us that the entanglement is maximal at large η for all θ (Figure 3). In the particular case when $\theta = \frac{\pi}{2}$, the purity is typically a ratio of a hyperbolic cosine function, which shows clearly that the purity is positive as it should be (Figure 4). The second configuration gives also a mixing dependence between the hyperbolic and sinusoidal functions where the corresponding plots (Figures 5 and 6) show some difference in the form with respect to the first one. In both cases, we notice that the numerators are always hyperbolic cosine of even η and denominators are also power of the function $\cosh \eta$.

The present paper is organized as follows. In section 2, we review the derivation of the solutions of the energy spectrum of two coupled harmonic oscillators [32]. These will be used to build the corresponding coherent states and therefore evaluate the purity function of the reduced matrix elements in section 3. The final form of purity function is subjected to different investigations where we underline its dependence to two physical parameters η and θ . In section 4, we evaluate the purity in terms of the number states after a series of transformation. Two interesting case of the purity will be discussed in section 5. Finally, we give conclusion and perspective of our work.

2 Energy spectrum solutions

In doing our task, we consider a system of two coupled harmonic oscillators parameterized by the planar coordinates (X_1, X_2) and masses (m_1, m_2) . Accordingly, the corresponding Hamiltonian is

written as the sum of free and interacting parts [33]

$$H_1 = \frac{1}{2m_1}P_1^2 + \frac{1}{2m_2}P_2^2 + \frac{1}{2}(C_1X_1^2 + C_2X_2^2 + C_3X_1X_2) \quad (1)$$

where C_1, C_2 and C_3 are constant parameters. After rescaling the position variables

$$x_1 = \mu X_1, \quad x_2 = \mu^{-1} X_2 \quad (2)$$

as well as the momenta

$$p_1 = \mu^{-1}P_1, \quad p_2 = \mu P_2 \quad (3)$$

H_1 can be written as

$$H_2 = \frac{1}{2m}(p_1^2 + p_2^2) + \frac{1}{2}(c_1x_1^2 + c_2x_2^2 + c_3x_1x_2) \quad (4)$$

where the parameters are given by

$$\mu = (m_1/m_2)^{1/4}, \quad m = (m_1m_2)^{1/2}, \quad c_1 = C_1\sqrt{\frac{m_2}{m_1}}, \quad c_2 = C_2\sqrt{\frac{m_1}{m_2}}, \quad c_3 = C_3. \quad (5)$$

As the Hamiltonian (4) involves an interacting term, a straightforward investigation of the basic features of the system is not easy. Nevertheless, we can simplify this situation by a transformation to new phase space variables

$$y_a = M_{ab}x_b, \quad \hat{p}_a = M_{ab}p_b \quad (6)$$

where the matrix

$$(M_{ab}) = \begin{pmatrix} \cos \frac{\theta}{2} & -\sin \frac{\theta}{2} \\ \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix}. \quad (7)$$

is a unitary rotation with the mixing angle θ . Inserting the mapping (6) into (4), one realizes that θ should satisfy the condition

$$\tan \theta = \frac{c_3}{c_2 - c_1} \quad (8)$$

to get a factorizing Hamiltonian

$$H_3 = \frac{1}{2m}(\hat{p}_1^2 + \hat{p}_2^2) + \frac{k}{2}(e^{2\eta}y_1^2 + e^{-2\eta}y_2^2) \quad (9)$$

where we have introduced two parameters

$$k = \sqrt{c_1c_2 - c_3^2/4}, \quad e^{2\eta} = \frac{c_1 + c_2 + \sqrt{(c_1 - c_2)^2 + c_3^2}}{2k} \quad (10)$$

under the reserve that the condition $4c_1c_2 > c_3^2$ must be fulfilled. The parameter η is actually measuring the strength of the coupling.

For later use, it is convenient to separate the Hamiltonian (9) into two commuting parts and then write H_3 as

$$H_3 = e^\eta \mathcal{H}_1 + e^{-\eta} \mathcal{H}_2 \quad (11)$$

where \mathcal{H}_1 and \mathcal{H}_2 are given by

$$\mathcal{H}_1 = \frac{1}{2m}e^{-\eta}\hat{p}_1^2 + \frac{k}{2}e^\eta y_1^2, \quad \mathcal{H}_2 = \frac{1}{2m}e^\eta\hat{p}_2^2 + \frac{k}{2}e^{-\eta}y_2^2. \quad (12)$$

One can see that the decoupled Hamiltonian

$$H_0 = \frac{1}{2m}\hat{p}_1^2 + \frac{k}{2}y_1^2 + \frac{1}{2m}\hat{p}_2^2 + \frac{k}{2}y_2^2 \quad (13)$$

is obtained for $\eta = 0$, which is equivalent to set $c_3 = 0$.

The Hamiltonian H_3 can simply be diagonalized by defining a set of annihilation and creation operators. These are

$$a_i = \sqrt{\frac{k}{2\hbar\omega}} e^{\frac{\varepsilon\eta}{2}} y_i + \frac{i}{\sqrt{2m\hbar\omega}} e^{-\frac{\varepsilon\eta}{2}} \hat{p}_i, \quad a_i^\dagger = \sqrt{\frac{k}{2\hbar\omega}} e^{\frac{\varepsilon\eta}{2}} y_i - \frac{i}{\sqrt{2m\hbar\omega}} e^{-\frac{\varepsilon\eta}{2}} \hat{p}_i \quad (14)$$

with the frequency

$$\omega = \sqrt{\frac{k}{m}} \quad (15)$$

and $\varepsilon = \pm 1$ for $i = 1, 2$, respectively. They satisfy the commutation relations

$$[a_i, a_j^\dagger] = \delta_{ij} \quad (16)$$

whereas other commutators vanish. Now we can map H_3 in terms of a_i and a_i^\dagger as

$$H_3 = \hbar\omega \left(e^\eta a_1^\dagger a_1 + e^{-\eta} a_2^\dagger a_2 + \cosh \eta \right). \quad (17)$$

To obtain the eigenstates and the eigenvalues, one solves the eigenequation

$$H_3 |n_1, n_2\rangle = \mathcal{E}_{n_1, n_2} |n_1, n_2\rangle \quad (18)$$

getting the states

$$|n_1, n_2\rangle = \frac{(a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2}}{\sqrt{n_1! n_2!}} |0, 0\rangle \quad (19)$$

as well as the energy spectrum

$$E_{3, n_1, n_2} = \hbar\omega \left(e^\eta n_1 + e^{-\eta} n_2 + \cosh \eta \right). \quad (20)$$

It is clear that these eigenvalues reduce to those of the decoupled harmonic oscillators, namely $\hbar\omega (n_1 + n_2 + 1)$. This shows clearly that the presence of the coupling parameter η will make difference and allow us to derive interesting results in the forthcoming analysis.

To show the correlation between variables, let us just focus on the ground state and write the corresponding wavefunction in y -representation. This is

$$\psi_0(\vec{y}) \equiv \langle y_1, y_2 | 0, 0 \rangle = \sqrt{\frac{m\omega}{\pi\hbar}} \exp \left\{ -\frac{m\omega}{2\hbar} (e^\eta y_1^2 + e^{-\eta} y_2^2) \right\} \quad (21)$$

which can easily be used to deduce the ground state wavefunction in terms of the variables (x_1, x_2) . Therefore, from the unitary representation we find

$$\begin{aligned} \psi_0(\vec{x}) &\equiv \langle x_1, x_2 | 0, 0 \rangle \\ &= \sqrt{\frac{m\omega}{\pi\hbar}} \exp \left\{ -\frac{m\omega}{2\hbar} \left[e^\eta \left(x_1 \cos \frac{\theta}{2} - x_2 \sin \frac{\theta}{2} \right)^2 + e^{-\eta} \left(x_1 \sin \frac{\theta}{2} + x_2 \cos \frac{\theta}{2} \right)^2 \right] \right\}. \end{aligned} \quad (22)$$

We notice that (21) is separable in terms of the variables y_1 and y_2 , which is not the case for (22) in terms of x_1 and x_2 . We close this part by claiming that the obtained results so far will be used to study the entanglement in the present system.

3 Entanglement in coherent states

As we claimed above, we implement our approach to study the entanglement of two coupled harmonic oscillators. Actually, it can be seen as another alternative method to recover the results obtained in [19] not only in a simpler way but also with less physical parameters of control. To start let us first introduce the coherent states corresponding to the eigenstates $|n_1, n_2\rangle$ given in (19). As usual, we can use the displacement operator to define the coherent states in terms of two complex numbers α and β . These are

$$|\alpha, \beta\rangle = D(a_1, \alpha)D(a_2, \beta)|0, 0\rangle \quad (23)$$

which gives the wavefunction

$$\Phi_{\alpha\beta}(y_1, y_2) = \left(\frac{\lambda_1 \lambda_2}{\pi}\right)^{1/2} \exp \left[-\frac{\lambda_1^2}{2} y_1^2 - \frac{|\alpha|^2}{2} - \frac{\alpha^2}{2} + \sqrt{2}\alpha \lambda_1 y_1 - \frac{\lambda_2^2}{2} y_2^2 - \frac{|\beta|^2}{2} - \frac{\beta^2}{2} + \sqrt{2}\beta \lambda_2 y_2 \right] \quad (24)$$

where we have set the quantities

$$\lambda_1 = e^{\frac{\eta}{2}} \left(\frac{mk}{\hbar^2}\right)^{1/4}, \quad \lambda_2 = e^{-\frac{\eta}{2}} \left(\frac{mk}{\hbar^2}\right)^{1/4}. \quad (25)$$

In terms of the original variables (X_1, X_2) , (24) reads as

$$\begin{aligned} \Phi_{\alpha\beta}(X_1, X_2) &= \left(\frac{\lambda_1 \lambda_2}{\pi}\right)^{1/2} \exp \left[-\frac{\lambda_1^2}{2} \left(\mu \cos \frac{\theta}{2} X_1 - \frac{1}{\mu} \sin \frac{\theta}{2} X_2 \right)^2 - \frac{\lambda_2^2}{2} \left(\mu \sin \frac{\theta}{2} X_1 + \frac{1}{\mu} \cos \frac{\theta}{2} X_2 \right)^2 \right] \\ &\times \exp \left[\sqrt{2}\alpha \lambda_1 \left(\mu \cos \frac{\theta}{2} X_1 - \frac{1}{\mu} \sin \frac{\theta}{2} X_2 \right) + \sqrt{2}\beta \lambda_2 \left(\mu \sin \frac{\theta}{2} X_1 + \frac{1}{\mu} \cos \frac{\theta}{2} X_2 \right) \right] \\ &\times \exp \left[-\frac{|\alpha|^2}{2} - \frac{|\beta|^2}{2} - \frac{\alpha^2}{2} - \frac{\beta^2}{2} \right]. \end{aligned} \quad (26)$$

As it is clearly shown in the wavefunction (26), the non-separability of the variables will play in crucial role in discussing the entanglement in the present system. This statement will be clarified later on when we will come to the analysis of the role of the involved parameters.

At this level we have set all ingredients to study the entanglement in the present system. All we need is to determine explicitly the purity function that is a trace of the density square corresponding to the obtained eigenstates. More precisely, we have

$$P = \text{Tr} \rho^2 \quad (27)$$

which in terms of the above coherent states reads as

$$P_{\alpha\beta} = \int dX_1 dX'_1 dX_2 dX'_2 \Phi_{\alpha\beta}(X_1, X_2) \Phi_{\alpha\beta}^*(X'_1, X_2) \Phi_{\alpha\beta}(X'_1, X'_2) \Phi_{\alpha\beta}^*(X_1, X'_2). \quad (28)$$

Upon substitution, we obtain the form

$$\begin{aligned} P_{\alpha\beta} &= \left(\frac{\lambda_1 \lambda_2}{\pi}\right)^2 \int dX_1 dX'_1 dX_2 dX'_2 e^{-\mu^2(\lambda_1^2 \cos^2 \frac{\theta}{2} + \lambda_2^2 \sin^2 \frac{\theta}{2})(X_1^2 + X_1'^2) - \frac{1}{\mu^2}(\lambda_1^2 \sin^2 \frac{\theta}{2} + \lambda_2^2 \cos^2 \frac{\theta}{2})(X_2^2 + X_2'^2)} \\ &\times e^{\frac{1}{2}(\lambda_1^2 - \lambda_2^2) \sin \theta (X'_1 X_2 + X_1 X_2 + X'_1 X'_2 + X_1 X'_2)} e^{2\mu \left(\frac{\alpha + \alpha^*}{\sqrt{2}} \lambda_1 \cos \frac{\theta}{2} + \frac{\beta + \beta^*}{\sqrt{2}} \lambda_2 \sin \frac{\theta}{2} \right) (X_1 + X'_1)} \\ &\times e^{-\frac{2}{\mu} \left(\frac{\alpha + \alpha^*}{\sqrt{2}} \lambda_1 \sin \frac{\theta}{2} - \frac{\beta + \beta^*}{\sqrt{2}} \lambda_2 \cos \frac{\theta}{2} \right) (X_2 + X'_2)} e^{-2|\alpha|^2 - 2|\beta|^2 - \alpha^2 - \alpha^{*2} - \beta^2 - \beta^{*2}}. \end{aligned} \quad (29)$$

This integral can easily be evaluated by introducing an appropriate transformation. This can be done by making use of the following change of variables

$$\begin{pmatrix} X_1 \\ X'_1 \\ X_2 \\ X'_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \frac{\omega_1}{\mu\sqrt{1-2a}} & \frac{\sqrt{2}}{\mu}\omega_1 & \frac{\omega_1}{\mu\sqrt{1+2a}} & 0 \\ \frac{\omega_1}{\mu\sqrt{1-2a}} & -\frac{\sqrt{2}}{\mu}\omega_1 & \frac{\omega_1}{\mu\sqrt{1+2a}} & 0 \\ -\frac{\mu\omega_2}{\sqrt{1-2a}} & 0 & \frac{\mu\omega_2}{\sqrt{1+2a}} & \sqrt{2}\mu\omega_2 \\ -\frac{\mu\omega_2}{\sqrt{1-2a}} & 0 & \frac{\mu\omega_2}{\sqrt{1+2a}} & -\sqrt{2}\mu\omega_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \quad (30)$$

where ω_1 , ω_2 and a are given by

$$\omega_1 = \frac{1}{\sqrt{\lambda_1^2 \cos^2 \frac{\theta}{2} + \lambda_2^2 \sin^2 \frac{\theta}{2}}}, \quad \omega_2 = \frac{1}{\sqrt{\lambda_1^2 \sin^2 \frac{\theta}{2} + \lambda_2^2 \cos^2 \frac{\theta}{2}}}, \quad a = -\frac{1}{4}(\lambda_1^2 - \lambda_2^2) \sin \theta \omega_1 \omega_2. \quad (31)$$

By showing that the determinant of such transformation is $\frac{\omega_1 \omega_2}{\lambda_1 \lambda_2}$, it is easy to map $P_{\alpha\beta}$ in terms of the new variables as

$$\begin{aligned} P_{\alpha\beta} &= \frac{1}{\pi^2} \lambda_1 \lambda_2 \omega_1 \omega_2 e^{-2|\alpha|^2 - 2|\beta|^2 - \alpha^2 - \alpha^{*2} - \beta^2 - \beta^{*2}} \int_{-\infty}^{+\infty} du_2 du_4 e^{-u_2^2 - u_4^2} \\ &\times \int_{-\infty}^{+\infty} du_1 e^{-u_1^2 + \frac{\sqrt{2}}{\sqrt{1-2a}} [\lambda_1 (\omega_1 \cos \frac{\theta}{2} + \omega_2 \sin \frac{\theta}{2})(\alpha + \alpha^*) + \lambda_2 (\omega_1 \sin \frac{\theta}{2} - \omega_2 \cos \frac{\theta}{2})(\beta + \beta^*)] u_1} \\ &\times \int_{-\infty}^{+\infty} du_3 e^{-u_3^2 + \frac{\sqrt{2}}{\sqrt{1+2a}} [\lambda_1 (\omega_1 \cos \frac{\theta}{2} - \omega_2 \sin \frac{\theta}{2})(\alpha + \alpha^*) + \lambda_2 (\omega_1 \sin \frac{\theta}{2} + \omega_2 \cos \frac{\theta}{2})(\beta + \beta^*)] u_3}. \end{aligned} \quad (32)$$

Performing the integration to end up with the result

$$P_{\alpha\beta}(\eta, \theta) = \frac{1}{\sqrt{2 \cosh 2\eta \sin^2 \frac{\theta}{2} \cos^2 \frac{\theta}{2} + \cos^4 \frac{\theta}{2} + \sin^4 \frac{\theta}{2}}}. \quad (33)$$

This is among the interesting results derived so far in the present work. Indeed, it shows clearly that the purity depends on the physical parameters (η, θ) rather than the complex displacements (α, β) and hereafter it will be denoted by $P(\eta, \theta)$. Furthermore, the obtained purity is two parameters dependent, which means that it can be controlled easily. If one requires the decoupling case ($\eta = 0$), $P(\eta, \theta)$ reduces to one as expected and therefore there is no entanglement.

To understand better the above results, we recall that the purity is related to linear entropy by the simple form

$$L = 1 - P \quad (34)$$

where P lies in the interval $[0, 1]$. Now let us proceed to plot the purity for a range of η and by considering $\theta \in [0, \pi]$. From Figure 1, it is clear that the purity, as function of η , is symmetric with respect to the decoupling case $\eta = 0$. It is maximal for $\eta = 0$, which really shows that the system is disentangled. After that it decreases rapidly to reach zero and indicates that the entanglement is maximal. More importantly, the purity becomes constant whenever θ takes the value zero or π . This behavior of the purity traced in below tell us that one can easily play with two parameters to control the degree of entanglement in the present system.

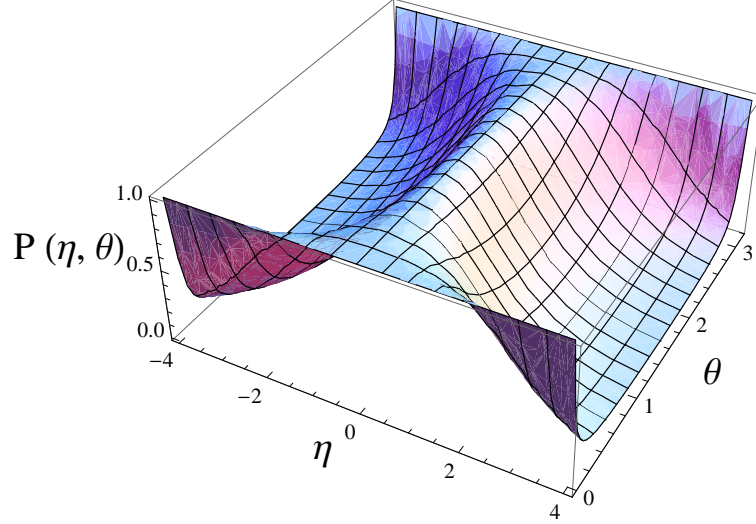


Figure 1: Purity in terms of the coupling parameter η and the mixing angle θ .

Specifically at $\theta = \frac{\pi}{2}$, we obtain a simple form

$$P\left(\eta, \theta = \frac{\pi}{2}\right) = \frac{1}{\cosh \eta} \quad (35)$$

which is one parameter dependent and can be adjusted only by varying the coupling η to control the degree of the entanglement. To be much more accurate, we underline such behavior by plotting (35) in Figure 2:

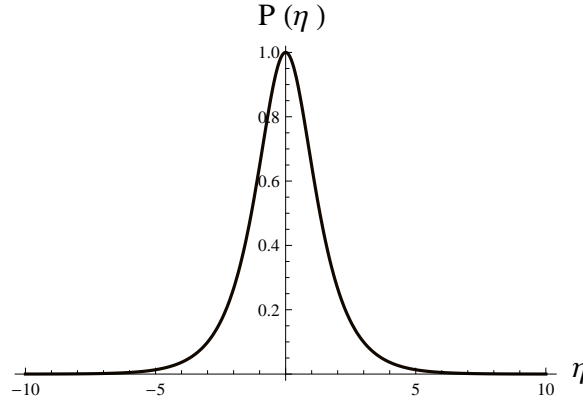


Figure 2: Purity in terms of the coupling parameter η for the mixing angle $\theta = \frac{\pi}{2}$.

From the above figure, one can deduce two interesting conclusions. The first one tells as that $P(\eta, \theta)$ is bounded, i.e. $0 \leq P \leq 1$, as expected. The second one shows clearly that the purity goes to zero for a strong coupling, which indicates the entanglement is maximal.

4 Entanglement in number states

To gain more information about the behavior of the present system, we evaluate the degree of the entanglement between inter states. For this, we consider the relation inverse to express the number states in terms of the coherent states. This is

$$|n_1, n_2\rangle = \frac{1}{\sqrt{n_1!n_2!}} \frac{\partial^{n_1}}{\partial \alpha^{n_1}} \frac{\partial^{n_2}}{\partial \beta^{n_2}} e^{|\alpha|^2} e^{|\beta|^2} |\alpha, \beta\rangle \Big|_{\alpha=0, \beta=0}. \quad (36)$$

In the y -representation, (36) leads to the wavefunction

$$\begin{aligned}\tilde{\Phi}_{n_1 n_2}(y_1, y_2) &= \tilde{\Phi}_{n_1 n_2} \left(\mu \cos \frac{\theta}{2} X_1 - \frac{1}{\mu} \sin \frac{\theta}{2} X_2, \mu \sin \frac{\theta}{2} X_1 + \frac{1}{\mu} \cos \frac{\theta}{2} X_2 \right) \equiv \tilde{\Phi}_{n_1 n_2}(X_1, X_2) \\ &= \frac{1}{\sqrt{n_1! n_2!}} \frac{\partial^{n_1}}{\partial \alpha^{n_1}} \frac{\partial^{n_2}}{\partial \beta^{n_2}} e^{\frac{|\alpha|^2}{2}} e^{\frac{|\beta|^2}{2}} \Phi_{\alpha\beta}(X_1, X_2) \Big|_{\alpha=0, \beta=0}\end{aligned}\quad (37)$$

where $\Phi_{\alpha\beta}(X_1, X_2)$ is given in (26). This will be implemented to study the purity in terms of the number states and discuss different issues.

Returning back to the purity definition, we have

$$P_{n_1 n_2} = \int dX_1 dX'_1 dX_2 dX'_2 \tilde{\Phi}_{n_1 n_2}(X_1, X_2) \tilde{\Phi}_{n_1 n_2}^*(X'_1, X'_2) \tilde{\Phi}_{n_1 n_2}(X'_1, X'_2) \tilde{\Phi}_{n_1 n_2}^*(X_1, X_2). \quad (38)$$

Using (37) to obtain the form

$$\begin{aligned}P_{n_1 n_2} &= \int dX_1 dX'_1 dX_2 dX'_2 \left(\frac{1}{n_1! n_2!} \right)^2 \\ &\times \frac{\partial^{n_1}}{\partial \alpha_1^{n_1}} \frac{\partial^{n_2}}{\partial \beta_1^{n_2}} e^{\frac{|\alpha_1|^2}{2}} e^{\frac{|\beta_1|^2}{2}} \Phi_{\alpha_1 \beta_1} \left(\mu \cos \frac{\theta}{2} X_1 - \frac{1}{\mu} \sin \frac{\theta}{2} X_2, \mu \sin \frac{\theta}{2} X_1 + \frac{1}{\mu} \cos \frac{\theta}{2} X_2 \right) \Big|_{\alpha_1, \beta_1=0} \\ &\times \frac{\partial^{n_1}}{\partial \alpha_2^{*n_1}} \frac{\partial^{n_2}}{\partial \beta_2^{*n_2}} e^{\frac{|\alpha_2|^2}{2}} e^{\frac{|\beta_2|^2}{2}} \Phi_{\alpha_2 \beta_2}^* \left(\mu \cos \frac{\theta}{2} X'_1 - \frac{1}{\mu} \sin \frac{\theta}{2} X'_2, \mu \sin \frac{\theta}{2} X'_1 + \frac{1}{\mu} \cos \frac{\theta}{2} X'_2 \right) \Big|_{\alpha_2^*, \beta_2^*=0} \\ &\times \frac{\partial^{n_1}}{\partial \alpha_3^{*n_1}} \frac{\partial^{n_2}}{\partial \beta_3^{*n_2}} e^{\frac{|\alpha_3|^2}{2}} e^{\frac{|\beta_3|^2}{2}} \Phi_{\alpha_3 \beta_3} \left(\mu \cos \frac{\theta}{2} X'_1 - \frac{1}{\mu} \sin \frac{\theta}{2} X'_2, \mu \sin \frac{\theta}{2} X'_1 + \frac{1}{\mu} \cos \frac{\theta}{2} X'_2 \right) \Big|_{\alpha_3, \beta_3=0} \\ &\times \frac{\partial^{n_1}}{\partial \alpha_4^{*n_1}} \frac{\partial^{n_2}}{\partial \beta_4^{*n_2}} e^{\frac{|\alpha_4|^2}{2}} e^{\frac{|\beta_4|^2}{2}} \Phi_{\alpha_4 \beta_4}^* \left(\mu \cos \frac{\theta}{2} x_1 - \frac{1}{\mu} \sin \frac{\theta}{2} x'_2, \mu \sin \frac{\theta}{2} X_1 + \frac{1}{\mu} \cos \frac{\theta}{2} X'_2 \right) \Big|_{\alpha_4^*, \beta_4^*=0}.\end{aligned}\quad (39)$$

After some algebra, we show that the purity takes the form

$$\begin{aligned}P_{n_1 n_2} &= \left(\frac{\lambda_1 \lambda_2}{\pi n_1! n_2!} \right)^2 \prod_{i=1}^4 \frac{\partial^{n_1}}{\partial \alpha_i^{n_1}} \frac{\partial^{n_2}}{\partial \beta_i^{n_2}} \int dX_1 dX'_1 dX_2 dX'_2 e^{-\frac{1}{2}(\alpha_1^2 + \beta_1^2 + \alpha_2^2 + \beta_2^2 + \alpha_3^2 + \beta_3^2 + \alpha_4^2 + \beta_4^2)} \\ &e^{-\frac{\lambda_1^2}{2} \left[\left(\mu \cos \frac{\theta}{2} X_1 - \frac{1}{\mu} \sin \frac{\theta}{2} X_2 \right)^2 + \left(\mu \cos \frac{\theta}{2} X'_1 - \frac{1}{\mu} \sin \frac{\theta}{2} X'_2 \right)^2 + \left(\mu \cos \frac{\theta}{2} X_1 - \frac{1}{\mu} \sin \frac{\theta}{2} X'_2 \right)^2 + \left(\mu \cos \frac{\theta}{2} X'_1 - \frac{1}{\mu} \sin \frac{\theta}{2} X_2 \right)^2 \right]} \\ &e^{-\frac{\lambda_2^2}{2} \left[\left(\mu \sin \frac{\theta}{2} X_1 + \frac{1}{\mu} \cos \frac{\theta}{2} X_2 \right)^2 + \left(\mu \sin \frac{\theta}{2} X'_1 + \frac{1}{\mu} \cos \frac{\theta}{2} X'_2 \right)^2 + \left(\mu \sin \frac{\theta}{2} X_1 + \frac{1}{\mu} \cos \frac{\theta}{2} X'_2 \right)^2 + \left(\mu \sin \frac{\theta}{2} X'_1 + \frac{1}{\mu} \cos \frac{\theta}{2} X_2 \right)^2 \right]} \\ &e^{\sqrt{2}\mu \left[(\lambda_1(\alpha_1 + \alpha_4) \cos \frac{\theta}{2} + \lambda_2(\beta_1 + \beta_4) \sin \frac{\theta}{2}) X_1 + (\lambda_1(\alpha_2 + \alpha_3) \cos \frac{\theta}{2} + \lambda_2(\beta_2 + \beta_3) \sin \frac{\theta}{2}) X'_1 \right]} \\ &e^{-\frac{\sqrt{2}}{\mu} \left[(\lambda_1(\alpha_1 + \alpha_2) \sin \frac{\theta}{2} - \lambda_2(\beta_2 + \beta_1) \cos \frac{\theta}{2}) X_2 + (\lambda_1(\alpha_3 + \alpha_4) \sin \frac{\theta}{2} - \lambda_2(\beta_3 + \beta_4) \cos \frac{\theta}{2}) X'_2 \right]}.\end{aligned}\quad (40)$$

This can be written, in a compact form, as

$$P_{n_1 n_2} = \left(\frac{\lambda_1 \lambda_2}{\pi n_1! n_2!} \right)^2 \prod_{i=1}^4 \frac{\partial^{n_1}}{\partial \alpha_i^{n_1}} \frac{\partial^{n_2}}{\partial \beta_i^{n_2}} \int d^4 Z e^{-Z^t \cdot A \cdot Z + B^t \cdot Z + C} \quad (41)$$

where $z^T = \begin{pmatrix} X_1 & X'_1 & X_2 & X'_2 \end{pmatrix}$, the matrix A is given by

$$A = \begin{pmatrix} A_{11} & 0 & A_{13} & A_{13} \\ 0 & A_{11} & A_{13} & A_{13} \\ A_{13} & A_{13} & A_{33} & 0 \\ A_{13} & A_{13} & 0 & A_{33} \end{pmatrix} \quad (42)$$

such that their components read as

$$A_{11} = \mu^2 \left(\lambda_1^2 \cos^2 \frac{\theta}{2} + \lambda_2^2 \sin^2 \frac{\theta}{2} \right), \quad A_{33} = \frac{1}{\mu} \left(\lambda_1^2 \sin^2 \frac{\theta}{2} + \lambda_2^2 \cos^2 \frac{\theta}{2} \right), \quad A_{13} = \frac{\lambda_2^2 - \lambda_1^2}{4} \sin \theta \quad (43)$$

and the matrix B takes the form

$$B = \sqrt{2} \begin{pmatrix} \mu\lambda_1(\alpha_1 + \alpha_4) \cos \frac{\theta}{2} + \mu\lambda_2(\beta_1 + \beta_4) \sin \frac{\theta}{2} \\ \mu\lambda_1(\alpha_2 + \alpha_3) \cos \frac{\theta}{2} + \mu\lambda_2(\beta_2 + \beta_3) \sin \frac{\theta}{2} \\ \frac{1}{\mu}\lambda_2(\beta_1 + \beta_2) \cos \frac{\theta}{2} - \frac{1}{\mu}\lambda_1(\alpha_1 + \alpha_2) \sin \frac{\theta}{2} \\ \frac{1}{\mu}\lambda_2(\beta_3 + \beta_4) \cos \frac{\theta}{2} - \frac{1}{\mu}\lambda_1(\alpha_3 + \alpha_4) \sin \frac{\theta}{2} \end{pmatrix}. \quad (44)$$

To go further in evaluating the purity, we perform a method to simplify our calculation. This can be done by introducing the change of variables

$$\begin{pmatrix} X_1 \\ X'_1 \\ X_2 \\ X'_2 \end{pmatrix} = \begin{pmatrix} \frac{\omega_1}{2\mu\sqrt{1-2a}} & \frac{\sqrt{2}}{2\mu}\omega_1 & \frac{\omega_1}{2\mu\sqrt{1+2a}} & 0 \\ \frac{\omega_1}{2\mu\sqrt{1-2a}} & -\frac{\sqrt{2}}{2\mu}\omega_1 & \frac{\omega_1}{2\mu\sqrt{1+2a}} & 0 \\ -\frac{\mu\omega_2}{2\sqrt{1-2a}} & 0 & \frac{\mu\omega_2}{2\sqrt{1+2a}} & \frac{\sqrt{2}}{2}\mu\omega_2 \\ -\frac{\mu\omega_2}{2\sqrt{1-2a}} & 0 & \frac{\mu\omega_2}{2\sqrt{1+2a}} & -\frac{\sqrt{2}}{2}\mu\omega_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \quad (45)$$

where the corresponding measure is $dX_1 dX'_1 dX_2 dX'_2 = J dx_1 dx_2 dx_3 dx_4$ and the Jacobian reads as

$$J = \frac{1}{\lambda_1 \lambda_2 \sqrt{(\lambda_1^2 \cos^2 \frac{\theta}{2} + \lambda_2^2 \sin^2 \frac{\theta}{2})(\lambda_1^2 \sin^2 \frac{\theta}{2} + \lambda_2^2 \cos^2 \frac{\theta}{2})}}. \quad (46)$$

This performance allows us to map (41) as

$$P_{n_1 n_2} = \left(\frac{\lambda_1 \lambda_2}{\pi n_1! n_2!} \right)^2 J \prod_{i=1}^4 \frac{\partial^{n_1}}{\partial \alpha_i^{n_1}} \frac{\partial^{n_2}}{\partial \beta_i^{n_2}} e^{-\frac{1}{2}(\alpha_i^2 + \beta_i^2)} \int d^4 Q e^{-Q^2 + D^t Q} \quad (47)$$

where $Q^t = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix}$ and D^t is the transpose of D , such as

$$D = \sqrt{2} \begin{pmatrix} \frac{\omega_1 \cos \frac{\theta}{2} + \omega_2 \sin \frac{\theta}{2}}{2\sqrt{1-2a}} \lambda_1(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) + \frac{\omega_1 \sin \frac{\theta}{2} - \omega_2 \cos \frac{\theta}{2}}{2\sqrt{1-2a}} \lambda_2(\beta_1 + \beta_2 + \beta_3 + \beta_4) \\ \frac{\sqrt{2}}{2} \omega_1 \lambda_1(\alpha_1 + \alpha_4 - \alpha_2 - \alpha_3) \cos \frac{\theta}{2} + \frac{\sqrt{2}}{2} \omega_1 \lambda_2(\beta_1 + \beta_4 - \beta_2 - \beta_3) \sin \frac{\theta}{2} \\ \frac{\omega_1 \cos \frac{\theta}{2} - \omega_2 \sin \frac{\theta}{2}}{2\sqrt{1+2a}} \lambda_1(\alpha_1 + \alpha_4 + \alpha_2 + \alpha_3) + \frac{\omega_1 \sin \frac{\theta}{2} + \omega_2 \cos \frac{\theta}{2}}{2\sqrt{1+2a}} \lambda_2(\beta_1 + \beta_4 + \beta_2 + \beta_3) \\ -\frac{\sqrt{2}}{2} \omega_2 \lambda_1(\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4) \sin \frac{\theta}{2} + \frac{\sqrt{2}}{2} \omega_2 \lambda_2(\beta_1 + \beta_2 - \beta_3 - \beta_4) \cos \frac{\theta}{2} \end{pmatrix}. \quad (48)$$

Since the above integral is Gaussian, then after some algebra we end with the form

$$\begin{aligned} P_{n_1 n_2} = & \left(\frac{\lambda_1 \lambda_2}{\pi n_1! n_2!} \right)^2 J \prod_{i=1}^4 \frac{\partial^{n_1}}{\partial \alpha_i^{n_1}} \frac{\partial^{n_2}}{\partial \beta_i^{n_2}} \exp \left[\frac{u}{\rho} \alpha_1^2 + \frac{2v}{\rho} \alpha_1 \alpha_2 - \frac{2u}{\rho} \alpha_1 \alpha_3 + \frac{2w}{\rho} \alpha_1 \alpha_4 + \frac{2s}{\rho} \alpha_1 \beta_1 \right. \\ & - \frac{2t}{\rho} \alpha_1 \beta_2 - \frac{2s}{\rho} \alpha_1 \beta_3 + \frac{2t}{\rho} \alpha_1 \beta_4 + \frac{u}{\rho} \alpha_2^2 + \frac{2w}{\rho} \alpha_2 \alpha_3 - \frac{2u}{\rho} \alpha_2 \alpha_4 - \frac{2t}{\rho} \alpha_2 \beta_1 + \frac{2s}{\rho} \alpha_2 \beta_2 \\ & + \frac{2t}{\rho} \alpha_2 \beta_3 - \frac{2s}{\rho} \alpha_2 \beta_4 + \frac{u}{\rho} \alpha_3^2 + \frac{2v}{\rho} \alpha_3 \alpha_4 - \frac{2s}{\rho} \alpha_3 \beta_1 + \frac{2t}{\rho} \alpha_3 \beta_2 + \frac{2s}{\rho} \alpha_3 \beta_3 - \frac{2t}{\rho} \alpha_3 \beta_4 \\ & + \frac{u}{\rho} \alpha_4^2 + \frac{2t}{\rho} \alpha_4 \beta_1 - \frac{2s}{\rho} \alpha_4 \beta_2 - \frac{2t}{\rho} \alpha_4 \beta_3 + \frac{2s}{\rho} \alpha_4 \beta_4 - \frac{u}{\rho} \beta_1^2 + \frac{2w}{\rho} \beta_1 \beta_2 \\ & \left. + \frac{2u}{\rho} \beta_1 \beta_3 + \frac{2v}{\rho} \beta_1 \beta_4 - \frac{u}{\rho} \beta_2^2 + \frac{2v}{\rho} \beta_2 \beta_3 + \frac{2u}{\rho} \beta_2 \beta_4 - \frac{u}{\rho} \beta_3^2 + \frac{2w}{\rho} \beta_3 \beta_4 - \frac{u}{\rho} \beta_4^2 \right] \end{aligned} \quad (49)$$

where we have set the involved parameters as

$$\begin{aligned}\rho &= 4\frac{mk}{\hbar^2} \left[2\cosh(2\eta) + \cot^2 \frac{\theta}{2} + \tan^2 \frac{\theta}{2} \right], & u &= \frac{2mk}{\hbar^2} \sinh 2\eta \\ v &= \frac{2mk}{\hbar^2} \left[\cosh(2\eta) + \tan^2 \frac{\theta}{2} \right], & w &= \frac{2mk}{\hbar^2} \left[\cosh(2\eta) + \cot^2 \frac{\theta}{2} \right] \\ t &= 4\frac{mk}{\hbar^2} \frac{\cosh \eta}{\sin \theta}, & s &= -4\frac{mk}{\hbar^2} \sinh \eta \frac{\cos \theta}{\sin \theta}.\end{aligned}\quad (50)$$

We are still looking for the final form of the purity, which can be obtained by calculating the partial derivatives. These can be performed in different ways and may be it is easier to proceed step by step. Indeed, we factorize the exponential function and then map each factor into a series expansion. This operation has been postponed to Appendix A and the yielded result is

$$P_{n_1 n_2}(\eta, \theta) = \frac{2\left(\frac{2}{\rho}\right)^{2(n_1+n_2)} (n_1! n_2!)^2}{\sin(\theta) \sqrt{2\cosh 2\eta + \tan^2\left(\frac{\theta}{2}\right) + \cot^2\left(\frac{\theta}{2}\right)}} \sum_{i+j+k+l+r=2(n_1+n_2)} C_{n_1 n_2}(i, j, k, l, r) u^i v^j w^k t^l s^r \quad (51)$$

where the coefficients $C_{n_1 n_2}$ are given by

$$\begin{aligned}C_{n_1 n_2} &= \frac{\left(\prod_{e=1}^4 \sum_{i_e=0}^{i_{e-1}}\right) \left(\prod_{e=1}^3 \sum_{j_e=0}^{j_{e-1}}\right) \left(\prod_{e=1}^3 \sum_{k_e=0}^{k_{e-1}}\right) \left(\prod_{e=1}^7 \sum_{l_e=0}^{l_{e-1}} \frac{1}{(l_{e-1} - l_e)!\right) \left(\prod_{e=1}^7 \sum_{r_e=0}^{r_{e-1}} \frac{1}{(r_{e-1} - r_e)!\right)} \\ &\quad \frac{2^{-i_4} (-1)^{l_1 - l_3 + l_4 - l_5 + l_6 - l_7 + r - r_1 + r_3 - r_5 + r_6 - r_7 + i_2 - c_1 - c_2}}{(i - i_1)! (i_1 - i_2)! (i_2 - i_3)! (i_3 - i_4)! l_7! r_7! c_3! c_4! c_5! c_6! c_7! c_8! c_9! c_{10}!}.\end{aligned}\quad (52)$$

It is clear that the final form of the purity is actually only depending on two parameters, i.e. η and θ . On the other hand, it is easy to check that $P_{n_1 n_2}$ is symmetric under the change of the quantum numbers n_1 and n_2 .

5 Two special cases

To be much more accurate let us illustrate some particular cases. With these we will be able to get more information from the above purity about the degree of entanglement. In the beginning, let us choose the configuration $(n_1 = 0, n_2 = 1)$, which means that we are considering now the entanglement between the ground state of the first oscillator and the first excited state of the second one. In this case, (51) reduces to the form

$$P_{01}(\eta, \theta) = \frac{2\left(\frac{2}{\rho}\right)^2}{\sin \theta \sqrt{2\cosh(2\eta) + \tan^2 \frac{\theta}{2} + \cot^2 \frac{\theta}{2}}} \sum_{l+r+j+k+i=2} C_{01}(i, j, k, l, r) u^i v^j w^k t^l s^r \quad (53)$$

which can be evaluated to obtain

$$P_{01}(\eta, \theta) = \frac{2\left(\frac{2}{\rho}\right)^2}{\sin \theta \sqrt{2\cosh(2\eta) + \tan^2 \frac{\theta}{2} + \cot^2 \frac{\theta}{2}}} (u^2 + v^2 + w^2) \quad (54)$$

and after replacing different parameters, one gets the final result

$$P_{01}(\eta, \theta) = \frac{3\cosh(4\eta) + 4\left(\tan^2 \frac{\theta}{2} + \cot^2 \frac{\theta}{2}\right)\cosh(2\eta) + 2\tan^4 \frac{\theta}{2} + 2\cot^4 \frac{\theta}{2} + 1}{\sin \theta \left(2\cosh(2\eta) + \tan^2 \frac{\theta}{2} + \cot^2 \frac{\theta}{2}\right)^{\frac{5}{2}}}. \quad (55)$$

This is a nice form, which can be worked more since it is only function of two physical parameters η and θ . Indeed, we plot it in Figure 3:

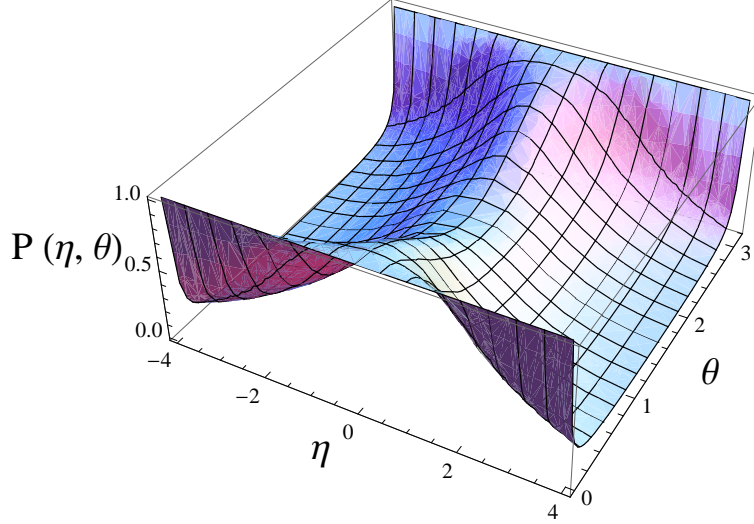


Figure 3: Purity P_{01} as function of the coupling parameter η and mixing angle θ for the quantum numbers $(n_1 = 0, n_2 = 1)$.

Here we have the same conclusion as in Figure 1 except that the present plot is showing some deformation at the point $\eta = 0$. Otherwise, for certain values of θ the purity is not always holding a maximum value at $\eta = 0$. More precisely, at this point it decreases to reach $1/2$ at $\theta = \frac{\pi}{2}$ and then increases to attends 1 at $\theta = \pi$. This is because in the present case the masses are equal and the same conclusion is obtained in [19].

Now let us look at some interesting situations by fixing the mixing angle θ and varying the coupling parameter η . In particular when $\theta = \frac{\pi}{2}$, P_{01} reduces to the form

$$P_{01} \left(\eta, \theta = \frac{\pi}{2} \right) = \frac{3 \cosh(4\eta) + 8 \cosh(2\eta) + 5}{32 \cosh^5 \eta}. \quad (56)$$

This can be plotted to obtain Figure 4:

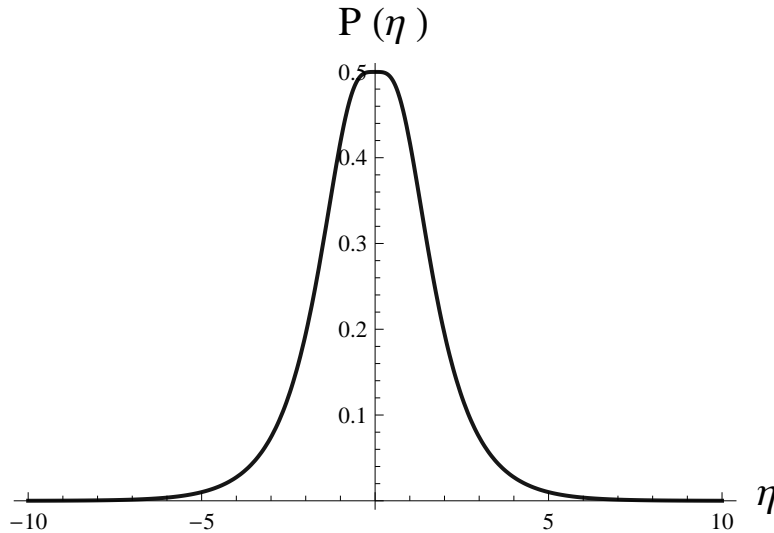


Figure 4: Purity P_{01} as function of η measuring the entanglement between the ground state $n_1 = 0$ and the first excited state $n_2 = 1$ for $\theta = \frac{\pi}{2}$.

Compared to Figure 2, we notice that the behavior of the purity in terms of the coupling parameter η is large. As long as η is large the entanglement is going to hold the maximum value. It shows clearly

the role playing by η and thus allows an easy control of the degree of the entanglement. This may give some hint about an experiment realization of the present case.

Now let us look at the case of the entanglement between the two first excited states of the two oscillators, i.e. $n_1 = n_2 = 1$. This result gives

$$P_{11}(\eta, \theta) = \frac{2\left(\frac{2}{\rho}\right)^4}{\sin \theta \sqrt{2 \cosh(2\eta) + \tan^2 \frac{\theta}{2} + \cot^2 \frac{\theta}{2}}} \sum_{i+j+k+l+r=4} C_{11}(i, j, k, l, r) u^i v^j w^k t^l s^r \quad (57)$$

after lengthy but simple calculations, we find

$$\begin{aligned} P_{11} = & \frac{2\left(\frac{2}{\rho}\right)^4}{\sin \theta \sqrt{2 \cosh(2\eta) + \tan^2 \frac{\theta}{2} + \cot^2 \frac{\theta}{2}}} \\ & \times \left(u^4 + v^4 + w^4 + 2^2 s^4 + 2^2 t^4 + 2^4 s^2 t^2 + 2u^2 v^2 + 2u^2 w^2 + 2v^2 w^2 + 2^4 ustv - 2^4 ustw \right. \\ & \left. - 2^4 u^2 s^2 - 2^4 u^2 t^2 - 2^4 t^2 w^2 - 2^4 s^2 w^2 - 2^4 t^2 v^2 - 2^4 s^2 v^2 + 2^3 vws^2 + 2^3 vwt^2 \right). \end{aligned} \quad (58)$$

Finally, we obtain

$$\begin{aligned} P_{11}(\eta, \theta) = & \frac{1}{4 \sin \theta \left[2 \cosh(2\eta) + \tan^2 \frac{\theta}{2} + \cot^2 \frac{\theta}{2} \right]^{\frac{9}{2}}} \left[9 \cosh(8\eta) + 16 \left(\tan^2 \frac{\theta}{2} + \cot^2 \frac{\theta}{2} \right) \cosh 6\eta \right. \\ & + \left(96 \tan^4 \frac{\theta}{2} + 96 \cot^4 \frac{\theta}{2} - 36 \right) \cosh(4\eta) + 240 \left(\tan^2 \frac{\theta}{2} + \cot^2 \frac{\theta}{2} \right) \cosh(2\eta) \\ & \left. + 8 \tan^8 \frac{\theta}{2} + 8 \cot^8 \frac{\theta}{2} - 64 \tan^4 \frac{\theta}{2} - 64 \cot^4 \frac{\theta}{2} + 459 \right]. \end{aligned} \quad (59)$$

Comparing this with (55), we notice that the numerator of both of them is containing a hyperbolic cosine function of a even number of coupling parameter η and the denominators are power of $\cosh \eta$. To go further, we plot (58) in Figure 5:

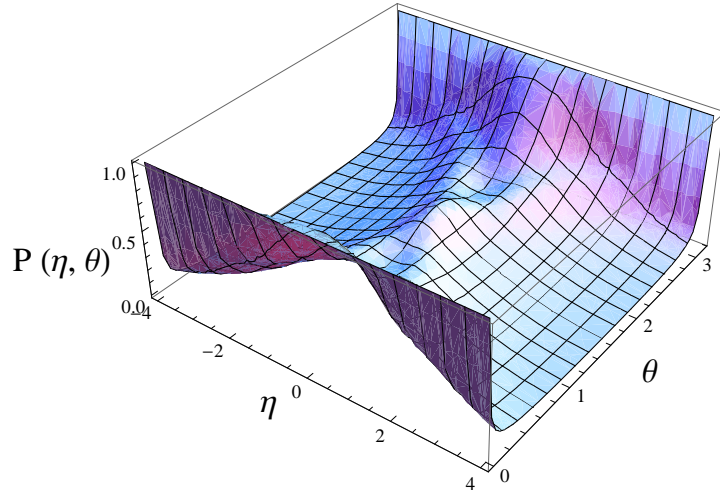


Figure 5: Purity P_{11} as function of the coupling parameter η and mixing angle θ for the quantum numbers ($n_1 = 1, n_2 = 1$).

Clearly, we see that for certain values of θ the purity is not always holding a maximum value at decoupling case, i.e $\eta = 0$. At this point, the purity decreases to reach $1/2$ at $\theta = \frac{\pi}{2}$ and then increases to attends 1 at $\theta = \pi$.

Furthermore, (59) can be worked much more to underline its behavior. The simplest way to do so is to fix the mixing angle θ and play with the coupling parameter η . For instance, by requiring $\theta = \frac{\pi}{2}$ we end up with the form

$$P_{11}(\eta, \theta) = \frac{9 \cosh(8\eta) + 32 \cosh(6\eta) + 156 \cosh(4\eta) + 480 \cosh(2\eta) + 347}{2048 \cosh^9 \eta}. \quad (60)$$

This shows clearly that $P_{11}(\eta, \theta)$ is one parameter dependent and therefore it can be manipulated easily. For more precision, we plot (60) in Figure 6:

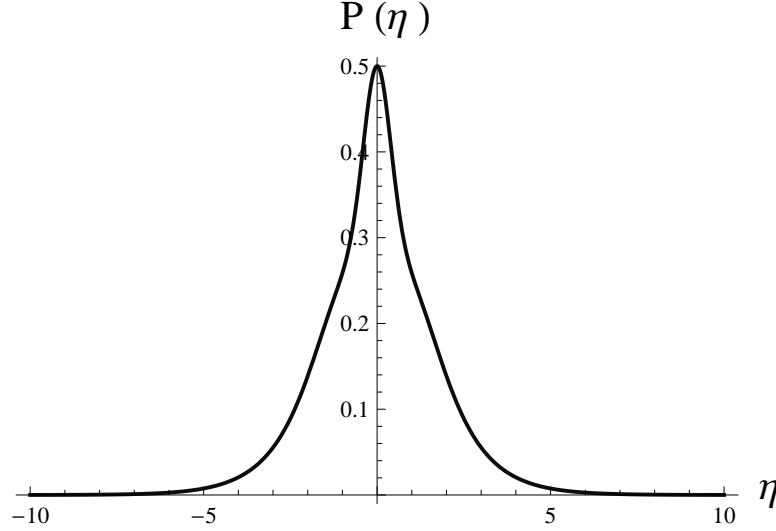


Figure 6: Purity P_{11} as function of η measuring the entanglement between the first exited states $(n_1 = 1, n_2 = 1)$ for $\theta = \frac{\pi}{2}$.

This is showing a difference with respect to Figure 4. It is clear that as long as η is small the purity increases rapidly to reach its maximal value. Also it decreases rapidly to attend zero for large η , which means that the system is strongly entangled.

6 Conclusion

The present work is devoted to study the entanglement of two coupled harmonic oscillators by adopting a new approach. For this, a Hamiltonian describing the system is considered and an unitary transformation is introduced. With this latter, the corresponding solutions of the energy spectrum are obtained in terms of the coupling parameter η and the mixing angle θ . It is clearly seen that when $\eta = 0$, the system becomes decoupled and therefore nothing new except harmonic oscillator in two dimensions.

To study the entanglement of the present system, we have introduced the purity function to evaluate its degree. In the beginning, we have realized the corresponding coherent states by using the standard method based on the displacement operator. These are used to determine explicitly the form of the purity in terms of the physical parameters η and θ . Also, the obtained result confirmed the range of the purity that is $0 \leq P \leq 1$. Moreover, we have clearly shown that purity is easy to control and can also be cast in a simple form when we fix $\theta = \frac{\pi}{2}$. In such case the purity is obtained as the

inverse of the hyperbolic function $\cosh \eta$ and the disentanglement simply corresponds to switching off η .

Subsequently, we have used the relation inverse between the number of states and the coherent states to determine the purity. After making different changes of variable, we have got a tractable Gaussian form, which was integrated easily. The final result showed that the purity is two parameters dependent. This allowed us to illustrate our finding by restricting ourselves to two particular cases. In the first configuration, we have considered the entanglement between the ground state and excited state, i.e. $(n_1 = 0, n_2 = 1)$ where the purity is exactly obtained. In the second configuration we studied the entanglement between the states $(n_1 = 1, n_2 = 1)$. In both cases, we have analyzed the case where $\theta = \frac{\pi}{2}$, which showed that a strong dependence of the purity to the hyperbolic cosine function of even coupling parameter.

On the other hand, the system of two coupled oscillators can serve as an analog computer for many of the physical theories and models. Therefore, one can extend the method developed here to study the entanglement in other interesting systems those illustrating the Feynman's rest. Furthermore, one immediate extension is to consider the case of a coupled systems submitted to an external magnetic field. This work and related matter are actually under consideration.

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Appendix A: Final form of purity

In this appendix, we show how to derive the final form of the purity given in (51). Indeed from (49), we obtain the result

$$\begin{aligned}
P_{n_1 n_2} = & \left(\frac{\lambda_1 \lambda_2}{\pi n_1! n_2!} \right)^2 J \sum_{i,j,k,l,r=0}^{\infty} \left(\prod_{e=1}^{11} \sum_{i_e=0}^{i_{e-1}} \frac{1}{(i_{e-1} - i_e)!} \right) \left(\prod_{e=1}^3 \left[\sum_{j_e=0}^{j_{e-1}} \frac{1}{(j_{e-1} - j_e)!} \sum_{k_e=0}^{k_{e-1}} \frac{1}{(k_{e-1} - k_e)!} \right] \right) \\
& \left(\prod_{e=1}^7 \left[\sum_{l_e=0}^{l_{e-1}} \frac{1}{(l_{e-1} - l_e)!} \sum_{r_e=0}^{r_{e-1}} \frac{1}{(r_{e-1} - r_e)!} \right] \right) \left(\frac{u}{\rho} \right)^i \left(\frac{2v}{\rho} \right)^j \left(\frac{2w}{\rho} \right)^k \left(\frac{2t}{\rho} \right)^l \left(\frac{2s}{\rho} \right)^r \frac{2^{i-i_4}}{i_{11}! j_3! k_3! l_7! r_7!} \\
& \left(\frac{\partial^{n_1}}{\partial \alpha_1^{n_1}} \alpha_1^{a_1} \right) \left(\frac{\partial^{n_1}}{\partial \alpha_2^{n_1}} \alpha_2^{a_2} \right) \left(\frac{\partial^{n_1}}{\partial \alpha_3^{n_1}} \alpha_3^{a_3} \right) \left(\frac{\partial^{n_1}}{\partial \alpha_4^{n_1}} \alpha_4^{a_4} \right) \left(\frac{\partial^{n_2}}{\partial \beta_1^{n_2}} \beta_1^{a_5} \right) \left(\frac{\partial^{n_2}}{\partial \beta_2^{n_2}} \beta_2^{a_6} \right) \\
& \left(\frac{\partial^{n_2}}{\partial \beta_3^{n_2}} \beta_3^{a_7} \right) \left(\frac{\partial^{n_2}}{\partial \beta_4^{n_2}} \beta_4^{a_8} \right) \Big|_{(\alpha_i, \beta_i) = (0,0)} \tag{A1}
\end{aligned}$$

where different parameters are given by

$$\begin{aligned}
a_1 &= 2i_{11} + i_3 - i_4 + l_1 - l_2 + l - l_1 + r_1 - r_2 + r - r_1 + j_3 + k - k_1 \\
a_2 &= 2i_{10} - 2i_{11} + i_2 - i_3 + l_7 + l_6 - l_7 + r_6 - r_7 + r_5 - r_6 + j_3 + k_3 \\
a_3 &= 2i_9 - 2i_{10} + i_3 - i_4 + l_5 - l_6 + l_4 - l_5 + r_7 + r_4 - r_5 + j_2 - j_3 + k_3 \\
a_4 &= i_2 - i_3 + 2i_8 - 2i_9 + j_2 - j_3 + l_3 - l_4 + l_2 - l_3 + k - k_1 + r_3 - r_4 + r_2 - r_3 \\
a_5 &= r_1 - r_2 + i_1 - i_2 + l_6 - l_7 + l_3 - l_4 + 2i_7 - 2i_8 + r_4 - r_5 + j_1 - j_2 + k_2 - k_3 \\
a_6 &= l_5 - l_6 + i - i_1 + l_1 - l_2 + 2i_6 - 2i_7 + r_5 - r_6 + r_3 - r_4 + j - j_1 + k_2 - k_3 \\
a_7 &= l_2 - l_3 + l_7 + i_1 - i_2 + r_7 + r - r_1 + j - j_1 + k_1 - k_2 + 2i_5 - 2i_6 \\
a_8 &= l - l_1 + l_4 - l_5 + i - i_1 + r_6 - r_7 + r_2 - r_3 + 2i_4 - 2i_5 + k_1 - k_2 + j_1 - j_2
\end{aligned}$$

and for the coherence of notations, $(i_0, j_0, k_0, l_0, r_0) \equiv (i, j, k, l, r)$ has to be under heard. Making use of the well-known formula

$$\left. \frac{\partial}{\partial x^n} x^l \right|_{x=0} = n! \delta_{l,n} \quad (\text{A2})$$

we end up with the form

$$\begin{aligned}
P_{n_1 n_2} &= \left(\frac{\lambda_1 \lambda_2}{\pi} n_1! n_2! \right)^2 J \sum_{i,j,k,l,r=0}^{\infty} \left(\prod_{e=1}^{11} \sum_{i_e=0}^{i_{e-1}} \frac{1}{(i_{e-1}-i_e)!} \right) \left(\prod_{e=1}^3 \sum_{j_e=0}^{j_{e-1}} \frac{1}{(j_{e-1}-j_e)!} \right) \left(\prod_{e=1}^3 \sum_{k_e=0}^{k_{e-1}} \frac{1}{(k_{e-1}-k_e)!} \right) \\
&\times \left(\prod_{e=1}^7 \sum_{l_e=0}^{l_{e-1}} \frac{1}{(l_{e-1}-l_e)!} \right) \left(\prod_{e=1}^7 \sum_{r_e=0}^{r_{e-1}} \frac{1}{(r_{e-1}-r_e)!} \right) \left(\frac{u}{\rho} \right)^i \left(\frac{2v}{\rho} \right)^j \left(\frac{2w}{\rho} \right)^k \left(\frac{2t}{\rho} \right)^l \left(\frac{2s}{\rho} \right)^r \\
&\times \frac{2^{i-i_4}}{i_{11}! j_3! k_3! l_7! r_7!} \delta_{b_1, n_1} \delta_{b_2, n_1} \delta_{b_3, n_1} \delta_{b_4, n_1} \delta_{b_5, n_2} \delta_{b_6, n_2} \delta_{b_7, n_2} \delta_{b_8, n_2}. \quad (\text{A3})
\end{aligned}$$

This shows clearly that a non vanishing purity should satisfy a set of constraint on different quantum numbers. These are

$$\left\{ \begin{array}{l}
b_1 - n_1 = 2i_{11} + i_3 - i_4 - l_2 + l - r_2 + r + j_3 + k - k_1 - n_1 = 0 \\
b_2 - n_1 = 2i_{10} - 2i_{11} + i_2 - i_3 + l_6 - r_7 + r_5 + j_3 + k_3 - n_1 = 0 \\
b_3 - n_1 = 2i_9 - 2i_{10} + i_3 - i_4 - l_6 + l_4 + r_7 + r_4 - r_5 + j_2 - j_3 + k_3 - n_1 = 0 \\
b_4 - n_1 = i_2 - i_3 + 2i_8 - 2i_9 + j_2 - j_3 - l_4 + l_2 + k - k_1 - r_4 + r_2 - n_1 = 0 \\
b_5 - n_2 = l_5 - l_6 + i - i_1 + l_1 - l_2 + 2i_6 - 2i_7 + r_5 - r_6 + r_3 - r_4 + j - j_1 + k_2 - k_3 - n_2 = 0 \\
b_6 - n_2 = l_2 - l_3 + l_7 + i_1 - i_2 + r_7 + r - r_1 + j - j_1 + k_1 - k_2 + 2i_5 - 2i_6 - n_2 = 0 \\
b_7 - n_2 = r_1 - r_2 + i_1 - i_2 + l_6 - l_7 + l_3 - l_4 + 2i_7 - 2i_8 + r_4 - r_5 + j_1 - j_2 + k_2 - k_3 - n_2 = 0 \\
b_8 - n_2 = l - l_1 + l_4 - l_5 + i - i_1 + r_6 - r_7 + r_2 - r_3 + 2i_4 - 2i_5 + k_1 - k_2 + j_1 - j_2 - n_2 = 0.
\end{array} \right. \quad (\text{A4})$$

We arrange the labels into two sets that we refer to them as the principals and secondary ones, respectively. The so-called secondary ones disappear upon summation of the 8 constraints and we get

$$i + j + k + l + r = 2(n_1 + n_2) \quad (\text{A5})$$

which is the constraint on the principal labels. The main result that emerges is that the purity is only depending on two parameters, such as

$$P_{n_1 n_2}(\eta, \theta) = \frac{2 \left(\frac{2}{\rho} \right)^{2(n_1 + n_2)} (n_1! n_2!)^2}{\sin \theta \sqrt{2 \cosh(2\eta) + \tan^2 \frac{\theta}{2} + \cot^2 \frac{\theta}{2}}} \sum_{i+j+k+l+r=2(n_1+n_2)} C_{n_1 n_2}(i, j, k, l, r) u^i v^j w^k t^l s^r. \quad (\text{A6})$$

The most important future of our result is that the function $C_{n_1 n_2}(i, j, k, l, r)$ can now be derived exactly for any n_1 and n_2 . This is

$$C_{n_1 n_2} = \left(\prod_{e=1}^{11} \sum_{i_e=0}^{i_{e-1}} \frac{1}{(i_{e-1} - i_e)!} \right) \left(\prod_{e=1}^3 \sum_{j_e=0}^{j_{e-1}} \frac{1}{(j_{e-1} - j_e)!} \right) \left(\prod_{e=1}^3 \sum_{k_e=0}^{k_{e-1}} \frac{1}{(k_{e-1} - k_e)!} \right) \left(\prod_{e=1}^7 \sum_{l_e=0}^{l_{e-1}} \frac{1}{(l_{e-1} - l_e)!} \right) \\ \times \left(\prod_{e=1}^7 \sum_{r_e=0}^{r_{e-1}} \frac{1}{(r_{e-1} - r_e)!} \right) \left(\frac{2^{-i_4} (-1)^{i_2 - i_8} (-1)^{r - r_1 + r_3 - r_5 + r_6 - r_7} (-1)^{l_1 - l_3 + l_4 - l_5 + l_6 - l_7}}{i_{11}! j_3! k_3! l_7! r_7!} \right). \quad (\text{A7})$$

Using the above constraints, we show that $C_{n_1 n_2}$ can be reduced to the form

$$C_{n_1 n_2} = \left(\prod_{e=1}^4 \sum_{i_e=0}^{i_{e-1}} \right) \left(\prod_{e=1}^3 \sum_{j_e=0}^{j_{e-1}} \right) \left(\prod_{e=1}^3 \sum_{k_e=0}^{k_{e-1}} \right) \left(\prod_{e=1}^7 \sum_{l_e=0}^{l_{e-1}} \frac{1}{(l_{e-1} - l_e)!} \right) \left(\prod_{e=1}^7 \sum_{r_e=0}^{r_{e-1}} \frac{1}{(r_{e-1} - r_e)!} \right) \\ \frac{2^{-i_4} (-1)^{l_1 - l_3 + l_4 - l_5 + l_6 - l_7 + r - r_1 + r_3 - r_5 + r_6 - r_7 + i_2 - c_1 - c_2}}{(i - i_1)! (i_1 - i_2)! (i_2 - i_3)! (i_3 - i_4)! l_7! r_7! c_3! c_4! c_5! c_6! c_7! c_8! c_9! c_{10}!} \quad (\text{A8})$$

where the involved parameters are fixed as

$$\begin{aligned} c_1 &= \frac{1}{2} [2n_1 - (i_2 - i_3) - (j_2 - j_3) - (k - k_1) - (l_2 - l_3) - (l_3 - l_4) - (r_2 - r_3) - (r_3 - r_4) \\ &\quad (i_3 - i_4) - (r_4 - r_5) - (j_2 - j_3) - (l_4 - l_5) - (l_5 - l_6) - r_7 - k_3] \\ c_2 &= \frac{1}{2} [n_1 - (i_2 - i_3) - (r_5 - r_6) - (r_6 - r_7) - l_6 - j_3 - k_3 + (i_3 - i_4) - (l - l_1) \\ &\quad (l_1 - l_2) - (r - r_1) - (r_1 - r_2) - j_3 - (k - k_1)] \\ c_3 &= \left(\frac{n_1 - (i_3 - i_4) - (r_4 - r_5) - (j_2 - j_3) - (l_4 - l_5) - (l_5 - l_6) - r_7 - k_3}{2} \right)! \\ c_4 &= \left(\frac{n_1 - (i_2 - i_3) - (r_5 - r_6) - (r_6 - r_7) - l_6 - j_3 - k_3}{2} \right)! \\ c_5 &= \left(\frac{n_1 - (i_3 - i_4) - (l - l_1) - (l_1 - l_2) - (r - r_1) - (r_1 - r_2) - j_3 - (k - k_1)}{2} \right)! \\ c_6 &= \left(\frac{n_1 - (i_2 - i_3) - (j_2 - j_3) - (k - k_1) - (l_2 - l_3) - (l_3 - l_4) - (r_2 - r_3) - (r_3 - r_4)}{2} \right)! \\ c_7 &= \left(\frac{n_2 - (r_1 - r_2) - (i_1 - i_2) - (l_6 - l_7) - (l_3 - l_4) - (r_4 - r_5) - (j_1 - j_2) - (k_2 - k_3)}{2} \right)! \\ c_8 &= \left(\frac{n_2 - (l_5 - l_6) - (i - i_1) - (l_1 - l_2) - (r_5 - r_6) - (r_3 - r_4) - (j - j_1) - (k_2 - k_3)}{2} \right)! \\ c_9 &= \left(\frac{n_2 - (l - l_1) - (l_4 - l_5) - (i - i_1) - (r_6 - r_7) - (r_2 - r_3) - (k_1 - k_2) - (j_1 - j_2)}{2} \right)! \\ c_{10} &= \left(\frac{n_2 - (l_2 - l_3) - (i_1 - i_2) - (r - r_1) - (j - j_1) - (k_1 - k_2) - r_7 - l_7}{2} \right)! \end{aligned}$$

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